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Application of Elasto-Plastic Finite Element Analysis to the Contact Problems of Solids

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ALTHOUGH numerous closed form solutions for contact problems of solids are available, they are limited to simple geometries and loading conditions. Advances in the finite element method have made possible almost revolutionary changes in analyses of this kind.[1-4] The concept of differential displacements was used widely in these analyses and was applied to problems of elastic-rigid body contact. A more sophisticated analysis was developed recently by Otte[5] to handle contact problems of two elastic bodies. His method was based on the concept of normal force and displacement compatibility at the contact surfaces and allows the inclusion of interface friction effects. His work has some disadvantages; it involves rather tedious formulations, is limited to elastic systems, and was not derived for curved boundaries.

The present method is based on a simple concept which involves the introduction of a very thin layer of elasto-plastic interface elements between the two contacting solids. The mechanical properties of these interface elements are selected in such a way that they represent a noncompressible fluid (ideally plastic) when transmitting compressive loadings and a gas-like material upon partial or complete separation of the contacted surfaces. No additional formulation or changes are required when using established elasto-plastic finite element analysis techniques. This approach was used successfully in simulating complicated contact-separation problems of a nuclear reactor fuel element.[6]

A case of two concentric cylinders is considered to demonstrate the application of this technique for contact problems. The inner cylinder is subjected to a contant pressure and a time-varying normal surface traction, $P_o(t)$ is applied to the outside surface of the system as shown in Fig. 1b. The two cylinders are in contact initially and the contact pressure reduces as $P_o(t)$ increases and the two cylinders will eventually separate when $P_o(t)$ reaches a certain level. The choice of this case study is partially motivated by existing closed form solutions available for checking the finite element results.

For plane strain conditions, the stress and displacement components in a hollow cylinder subject to both internal pressure and external traction P_o as shown in Fig. 1a, can be calculated from the formula[7]

$$\sigma_{rr} = \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} - \frac{(P_i - P_o) a^2 b^2}{(b^2 - a^2) r^2} \tag{1a}$$

$$\sigma_{\theta\theta} = \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} + \frac{(P_i - P_o) a^2 b^2}{b^2 - a^2} \frac{1}{r^2} \tag{1b}$$

$$u_r = \frac{1-\nu}{E} \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} r + \frac{1+\nu}{E} \frac{a^2 b^2 (P_i - P_o)}{b^2 - a^2} \frac{1}{r} \tag{1c}$$

The contact pressure at the interface of this two cylinder system shown in Fig. 1b can be derived from the compatibility condition of radial displacement and produces

$$P_c = \frac{\dfrac{2a^2}{E_1(b^2-a^2)} P_i - \dfrac{2c^2}{E_2(c^2-b^2)} P_o}{\dfrac{1}{E_1}\left(\dfrac{b^2+a^2}{b^2-a^2} - \nu_1\right) + \dfrac{1}{E_2}\left(\dfrac{c^2+b^2}{c^2-b^2} + \nu_2\right)} \tag{2}$$

Where E is the modulus of elasticity, ν is the Poisson's ratio while the subscripts 1 and 2 designate the material properties of inner and outer cylinder, respectively.

The dimensions of the cylinders used for the numerical illustration are given in Fig. 1c. The pressure loadings are assumed to be $P_i = 74.14$ psi and $(dP_o/dt)(t) = 4$ psi/sec. The finite-element idealization for the fixed ends (plane strain) case is shown in Fig. 1c. The shaded element (no. 9) is the interface element which functions as previously described. Triangular elements were used near the interface for the better radial resolution of stresses in this region before separation takes place. A total of three stacks of elements were used in the longitudinal direction for examining the sliding of the interface as shown in Fig. 2. The two cylinders may "slide" in this simple case when contact pressure disappears. At this instant the finite element code would change material properties of the interface element as indicated in Table 1. Figure 2 shows the realtive contraction of the contact surfaces at various distances away from the fixed end. It may be noticed that relative sliding of the surfaces increases with distance

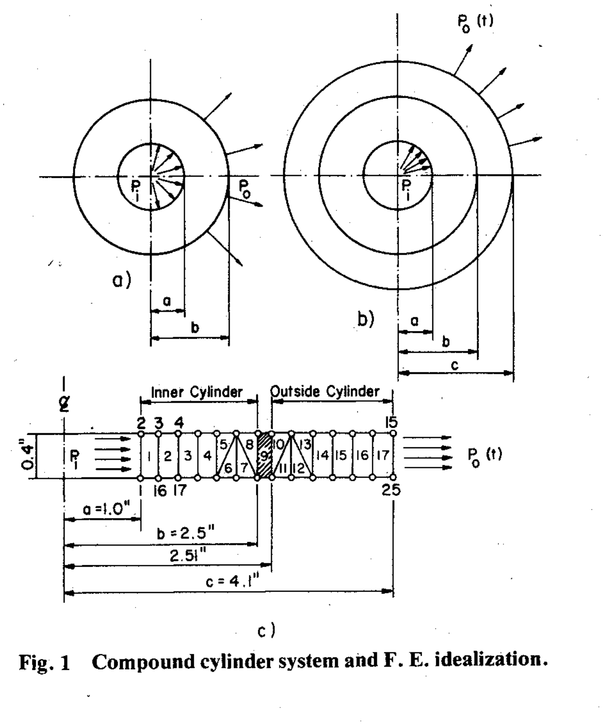

Fig. 1 Compound cylinder system and F. E. idealization.

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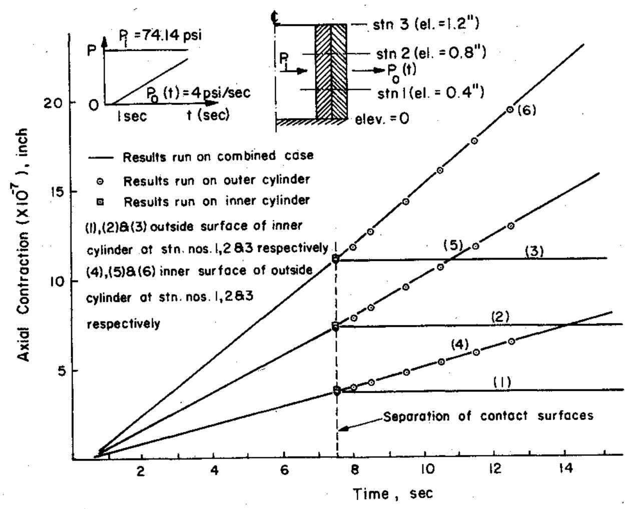

Fig. 2 Axial displacement at contact surfaces of two concentric cylinders.

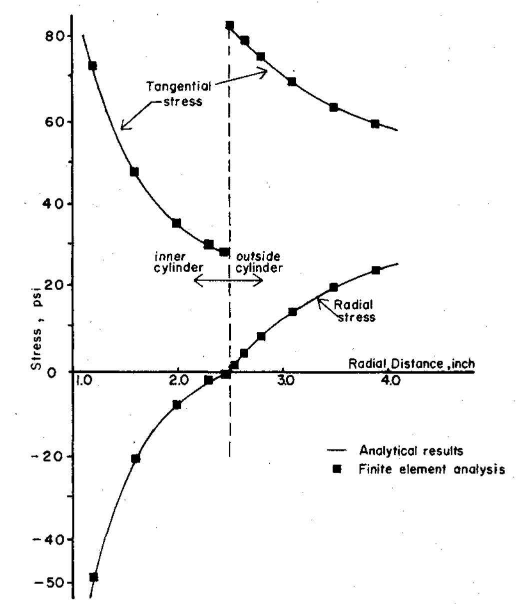

Fig. 3 Distribution of stresses in compound cylinder after separation.

from the end constraint. The various stress components in the cylinder system at separation are shown in Fig. 3.

The previous demonstration shows that in order to transmit pressure between two deforming solids in contact, the present method uses the idea of noncompressibility as exhibited by perfectly plastic interface elements capable of maintaining a hydrostatic condition in the interface. The same interface elements may also handle the separation of these solids with the appropriate switching of the element material properties from perfect plasticity to a low modulus of elasticity as indicated in Table 1. This technique is simple in principle and easy to apply. Its feasibility and applicability have been demonstrated with appropriate examples.

Another major advantage of this technique is its flexibility and potential for handling thermomechanical and inelastic contact problems. As mentioned earlier, the complicated thermomechanical contact behavior of a typical multi-material nuclear reactor fuel element has been satisfactorily simulated.[6]

The present method is limited to the frictionless sliding of contacting surfaces because of the lack of friction factors for practical structures. Such forces could be accommodated in the present scheme for example by varying interface element

Table 1 Material properties

	Inner cylinder	Outside cylinder	Interface layer: Before separation	Interface layer: After separation
Modulus of elasticity (E), psi	10^7	30×10^6	10^7	10^{-3}
Shear modulus of elasticity (G), psi	3.76×10^6	11.28×10^6	3.76×10^6	3.76×10^{-4}
Poisson's ratio	0.33	0.33	0.49	0.01
Yield strength, psi	30,000	90,000	1.0	10^7
Plastic modulus (E'), psi	0	0	0	0

plasticity properties. The plastic behavior of this material should be related to the coefficient of friction of the contacting solids. This procedure would require considerable effort in formulation but with the unique advantages already indicated above, the effort could well be worthwhile.

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Static Deflection of Beams Subjected to Elastic Rotational Restraints

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Introduction

THE beam shown in Fig. 1 is simply supported and elastically restrained against rotation at both ends. The deflection shape caused by a general case of loading may be obtained by successively integrating the differential equation

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